UCD Mathematical Enrichment 19/3/16

Geometry

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Problem 1

Lines PA and PB are tangent to a circle centred at O at points A and B respectively. A third tangent is drawn. It crosses PA and PB at X and Y respectively. Prove that $|\angle XOY|$ does not depend on the choice of the third tangent.

Problem 2

Let the circles $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect at A and B. Prove that the line AB is the radical axis of $\mathscr{C}_1$ and $\mathscr{C}_2$.

Problem 3

The incircle of ABC is tangent to side BC at K and an excircle is tangent to BC at L. Prove that $|CK| = |CL| = \frac{1}{2}(a + b - c)$, where a, b and c are the lengths of the sides of the triangle opposite A, B and C respectively.

Problem 4

Let the circles $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect at A and B. The line l_1, through a point P on $[AB]$, intersects $\mathscr{C}_1$ at K and L and the line l_2 through P intersects $\mathscr{C}_2$ at M and N. Prove that $KLMN$ is cyclic.

Problem 5

Prove that the power of a point P with respect to a circle $\mathscr{C}_1$ equals $d^2 - R^2$, where d is the distance from P to the centre of $\mathscr{C}_1$ and R is the radius of $\mathscr{C}_1$.

Problem 6

Prove that the radical axis of two circles bisects their common tangents.

Problem 7 (IrMO 2012)

A, B, C and D are four points in that order on the circumference of the circle K. AB is perpendicular to BC and BC is perpendicular to CD. X is a point on the circumference of the circle between A and D. AX extended meets CD extended at E and DX extended meets BA extended at F. Prove that the circumcircle of triangle AXF is tangent to the circumcircle of triangle DXE and that their common tangent line passes through the centre of the circle K.

Problem 8 (BMO 2013/2014 Round 1)

In the acute angled triangle ABC, the foot of the perpendicular from B to CA is E. Let l be the tangent to the circle ABC at B. The foot of the perpendicular from C to l is F. Prove that EF is parallel to AB.

Problem 9 (IMO 2015)

Triangle ABC has circumcircle Ω and circucentre O. A circle Γ with centre A intersects the segment BC at points D and E, such that B, D, E and C are all different and lie on line BC in this order. Let F and G be the points of intersection of Γ and Ω, such that A, F, B, C and G lie on Ω in this order. Let K be the second point of intersection of the circumcircle of triangle BDF and the segment AB. Let L be the second point of intersection of the circumcircle of triangle CGE and the segment CA.

Suppose that the lines FK and GL are different and intersect at the point X. Prove that X lies on the line AO.

Problem 10 (BMO 2004/2005 Round 1)

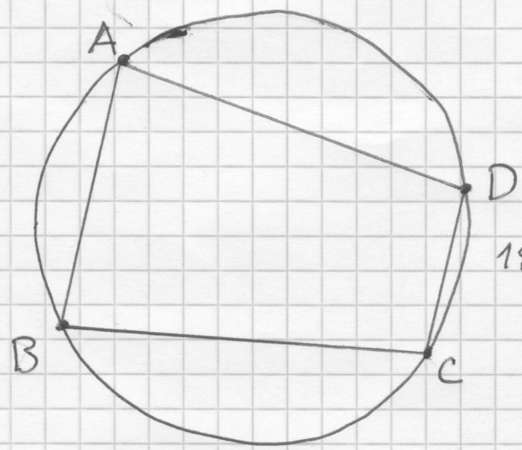
Let ABC be an acute angled triangle, and let D, E be the feet of the perpendiculars from A, B to BC, CA respectively. Let P be the point where the line AD meets the semicircle constructed outwardly on BC, and Q be the point where the line BE meets the semicircle constructed outwardly on AC. Prove that $CP = CQ$.

CIRCLES

First ...

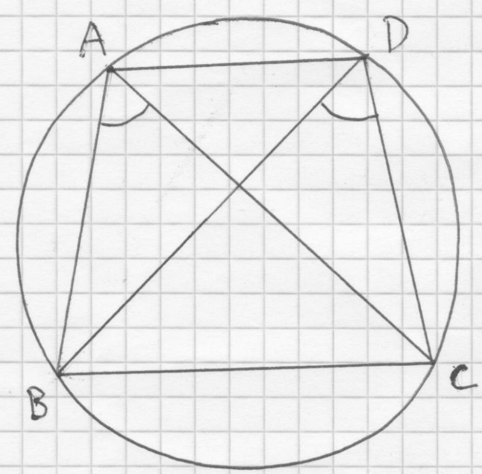
I Some basic theorems

1.) On cyclic quadrilaterals



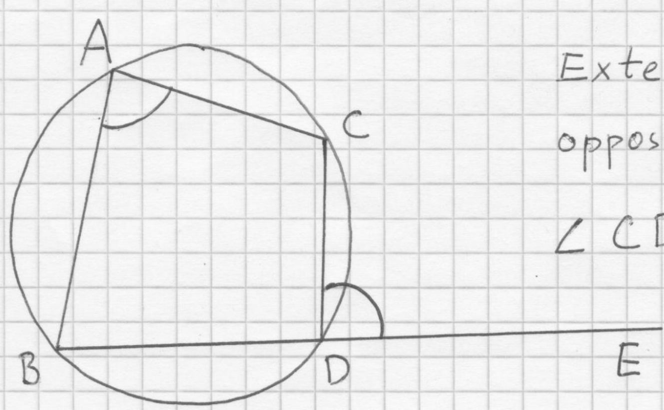
Opposite angles add up to 180°.

$180° = \angle DAB + \angle BCD = \angle ABC + \angle CDA$



Angles standing on the same arc are equal.

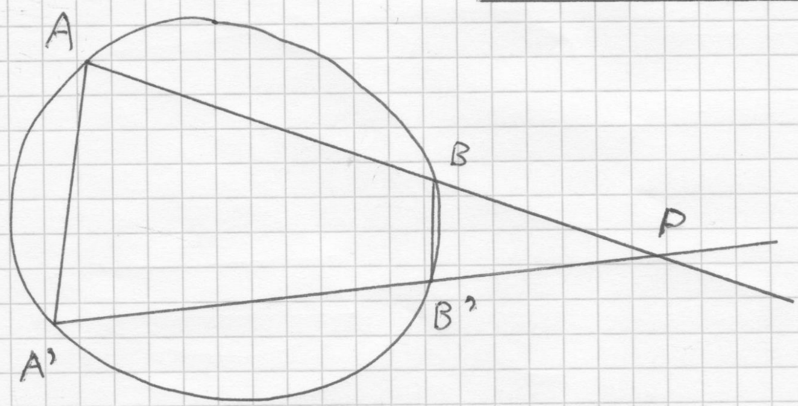
$\angle CAB = \angle CDB$

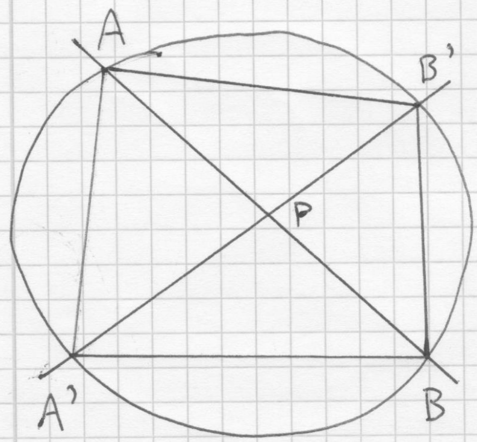


Exterior angle equals opposite interior angle.

$\angle CDE = \angle CAB$

Power of a point

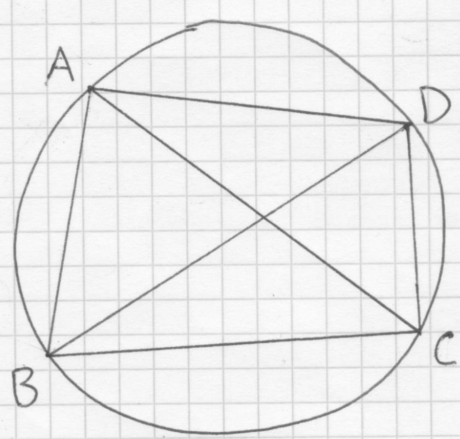




$|AP|.|BP| = |A'P|.|B'P|$

This will be covered in more detail later...

More obscure: <u>Ptolemy's Theorem</u>



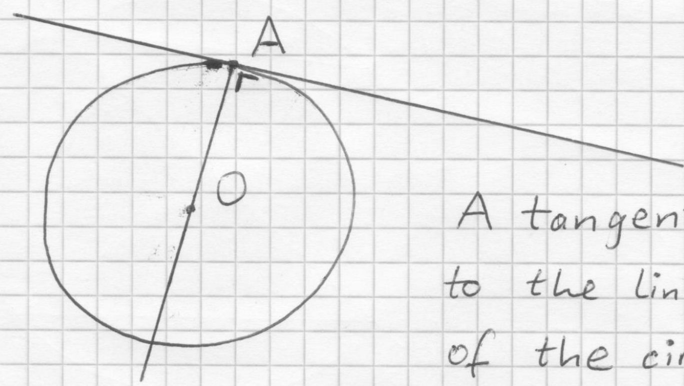
The sum of the products of the opposite sides equals the product of the diagonals.

$|AB|.|CD| + |AD|.|BC| = |AC|.|BD|$

All these conditions are <u>necessary</u> and <u>sufficient</u> for a quadrilateral to be cyclic. So if you show that one of them is true, all of them will be true.

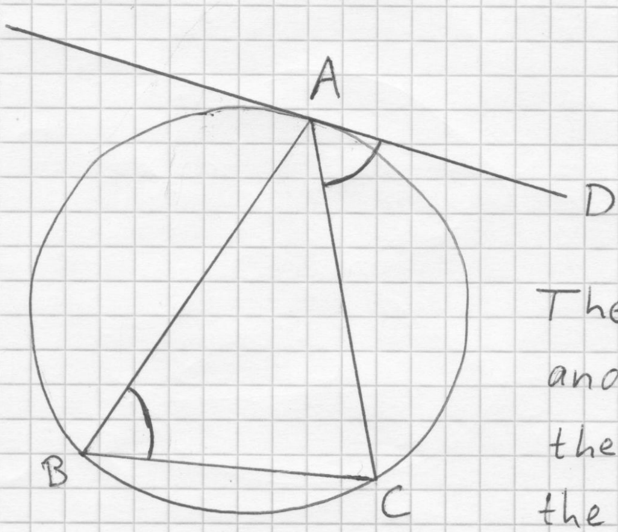
[NOTE: spotting a cyclic quadrilateral can be a key step to solving a problem]

2.) <u>On tangents to circles</u>



A tangent is perpendicular to the line through the centre of the circle which crosses it at the point of tangency.

Which is a special case of...



The angle between a tangent and a chord passing through the point of tangency equals the angle standing on the arc delimitted by the chord.

$\angle DAC = \angle ABC$

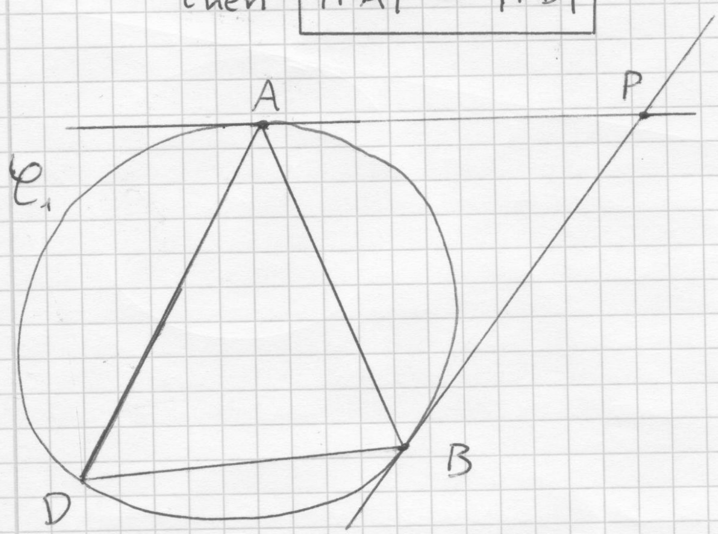
The last theorem will help us prove our first main result...

II Power of a Point

1.) The tangents through a point are equal

Let $\mathcal{C}_1$ be a circle and P a point outside $\mathcal{C}_1$. Let the tangents of $\mathcal{C}_1$ through P touch $\mathcal{C}_1$ at A and B,

then $\boxed{|PA| = |PB|}$



Proof:

Consider any point D on $\mathcal{C}_1$ for which $ABDP$ is a convex quadrilateral. Since PA is tangent to $\mathcal{C}_1$ and $\angle ADB$ stands on the chord AB,

$$\angle PAB = \angle ADB$$

Similarly, since PB is also a tangent to $\mathcal{C}_1$,

$$\angle PBA = \angle BDA$$

$$\therefore \angle PAB = \angle PBA$$

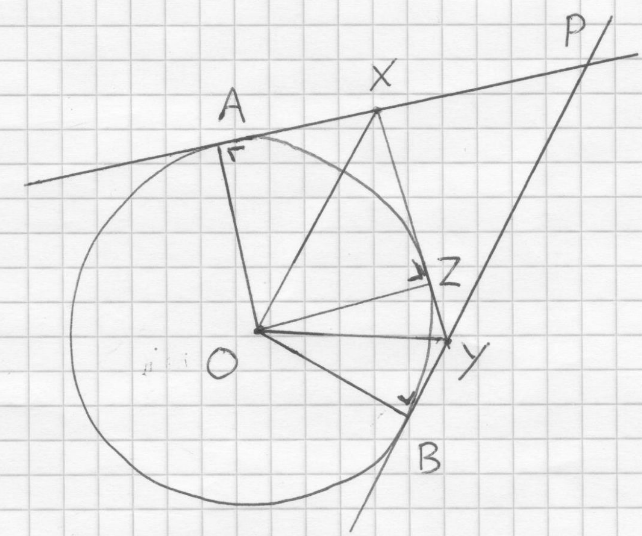
So the triangle ΔABP is isosceles.

Therefore $|PA| = |PB|$

[NOTE: You will often use facts about angles to prove statements about lengths and vice versa]

This result may seem simple but it is very powerful and can be used to prove more advanced theorems.

Problem 1: Lines PA and PB are tangent to a circle centred at O, at points A and B respectively. A third tangent is drawn. It crosses PA and PB at X and Y respectively. Prove that $|\angle XOY|$ does not depend on the choice of the third tangent.



Solution:

Let Z be the point of tangency of the third tangent. Consider the triangles ΔXAO and ΔXOZ. Since XA and XZ are both tangent to the circle, $|XA| = |XZ|$. Since AO and OZ are both radii of the circle, $|AO| = |OZ|$.

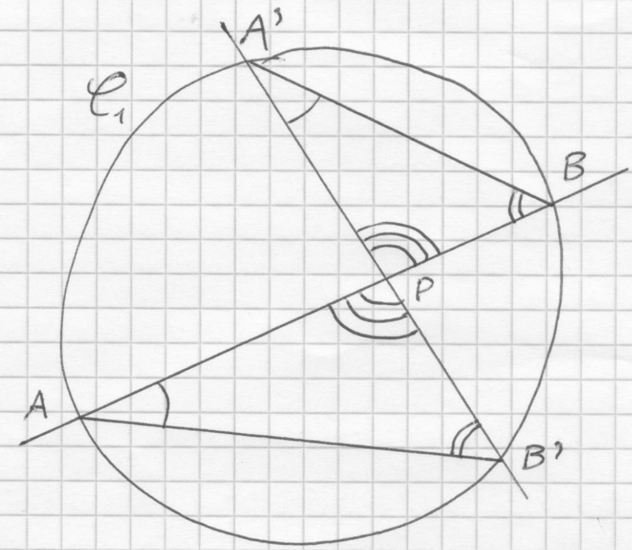
The segment XO is shared by both triangles. Therefore $\triangle XAO$ and $\triangle XOZ$ are congruent.
Therefore $|\angle AOX| = |\angle XOZ|$
By similar reasoning it can be shown that
$$|\angle ZOY| = |\angle YOB|$$
Since $|\angle XOY| = |\angle XOZ| + |\angle ZOY|$, we get
$$\begin{aligned} 2|\angle XOY| &= |\angle XOZ| + |\angle XOZ| + |\angle ZOY| + |\angle ZOY| \\ &= |\angle XOZ| + |\angle AOX| + |\angle ZOY| + |\angle YOB| \\ &= |\angle AOZ| + |\angle ZOB| \\ &= |\angle AOB| \end{aligned}$$
$$\Rightarrow |\angle XOY| = \tfrac{1}{2}|\angle AOB|$$
Since $|\angle AOB|$ does not depend on the choice of the third tangent, $|\angle XOY|$ does not depend on the choice of the third tangent.

2.) The power of a point

The property of tangents is just a special case of the power of the point P with respect to the circle $\mathcal{C}_1$.

Definition:
Let $\mathcal{C}_1$ be a circle and P a point (inside, outside or on the circle). Let ℓ be a line passing through P and crossing the circle at A and B. The quantity $|PA|\cdot|PB|$ is called the power of P with respect to $\mathcal{C}_1$ and does not depend on the choice of ℓ (as long as it passes through P and crosses $\mathcal{C}_1$ at least once).

We will prove the case when P lies inside $\mathcal{C}_1$ here. The proof for the other cases is similar and is left as an exercise.



Let two distinct lines through P cross the circle $\mathcal{C}_1$ at A, B and A', B' respectively. Note that the angles $\angle BAB'$ and $\angle BA'B'$ stand on the same arc, so $|\angle BAB'| = |\angle BA'B'|$. Similarly, $\angle A'BA$ and $\angle A'B'A$ stand on the same arc, so $|\angle A'BA| = |\angle A'B'A|$. Also note that $|\angle A'PB| = |\angle APB'|$ because these angles are symmetrical opposites. Therefore the triangles $\Delta A'BP$ and $\Delta AB'P$ are similar.

$$\Rightarrow \quad \frac{|PA'|}{|PA|} = \frac{|PB|}{|PB'|}$$

$$\Rightarrow \quad |PA|.|PB| = |PA'|.|PB'|$$

Now if we draw a third line through P and let it cross $\mathcal{C}_1$ at A'' and B'' we can repeat this procedure to show that

$$|PA|.|PB| = |PA''|.|PB''|$$

and so on. Therefore this relationship must hold regardless of our choice of lines.

Remarks:

- If P lies on $\mathcal{C}_1$, $|PA|\cdot|PB|=0$. This case is rarely used.
- If P lies outside $\mathcal{C}_1$ and ℓ is a tangent, $A=B$ and $|PA|\cdot|PB|=|PA|^2$

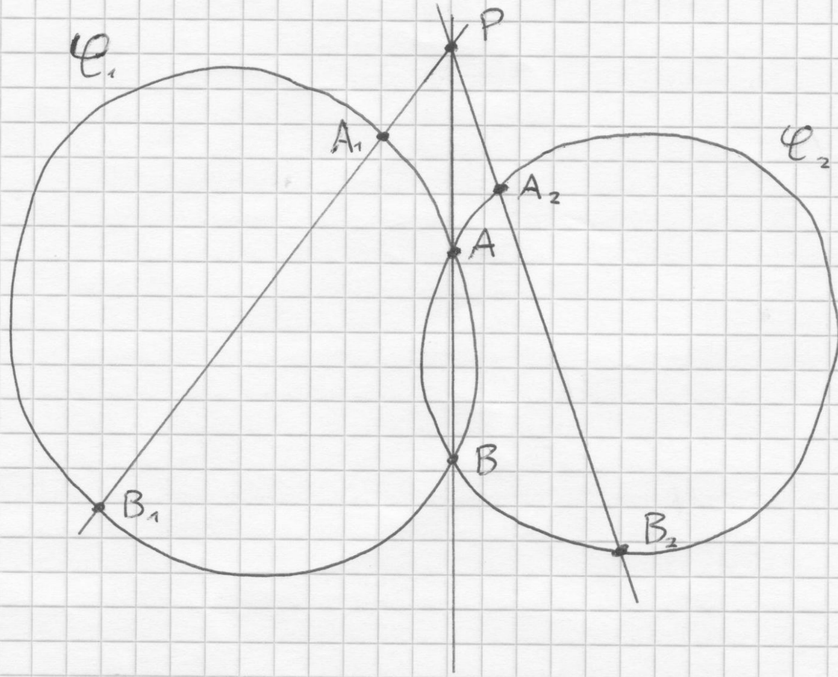
The power of a point is simple but powerful. It can be used to solve many difficult problems, including IMO level.

III The radical axis

Say we are given two circles, $\mathcal{C}_1$ and $\mathcal{C}_2$. Is the power of a point P ever equal for $\mathcal{C}_1$ and $\mathcal{C}_2$?

Definition:
Consider two circles $\mathcal{C}_1$ and $\mathcal{C}_2$. The radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$ is the line ℓ for which if a point P is on ℓ then the power of P with respect to $\mathcal{C}_1$ equals the power of P with respect to $\mathcal{C}_2$.

Problem 2: Let the circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. Prove that the line AB is the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$.



Consider any point P on AB. If we construct any line through P which intersects $\mathcal{C}_1$ and we label the points of intersection A_1 and B_1, then we have

$$|PA|.|PB| = |PA_1|.|PB_1|$$

Similarly, if we construct any line through P which intersects $\mathcal{C}_2$ at A_2 and B_2, we will have

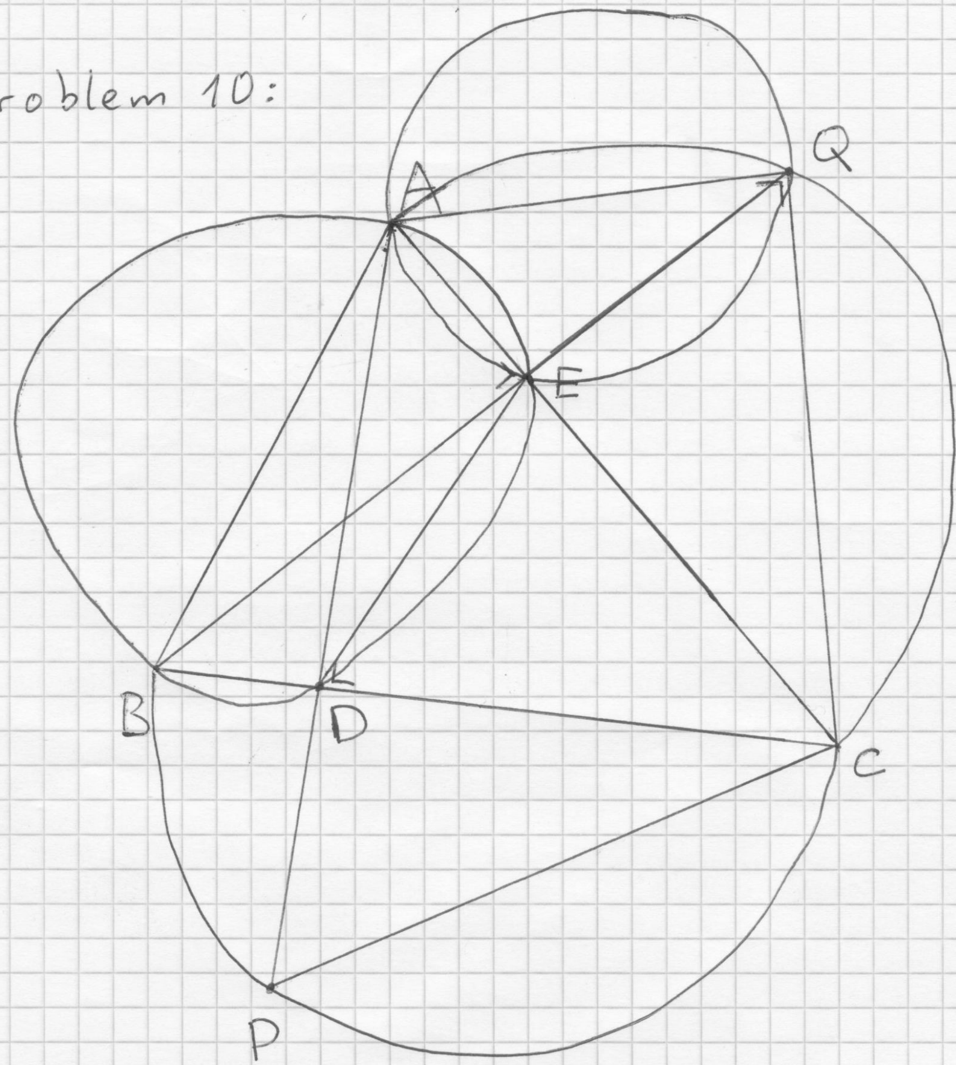
$$|PA|.|PB| = |PA_2|.|PB_2|$$

Therefore for any lines through P intersecting $\mathcal{C}_1$ and $\mathcal{C}_2$ at A_1, B_1 and A_2, B_2 respectively we have

$$|PA_1|.|PB_1| = |PA_2|.|PB_2|$$

So for any point P on AB, its power with respect to $\mathcal{C}_1$ equals its power with respect to $\mathcal{C}_2$. Therefore AB is the radical axis of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Problem 10:



Since the arc AQC is a semicircle,

$$|\angle AQC| = 90^\circ$$

Also $BE \perp AC$, so

$$|\angle AEQ| = 90^\circ$$

Therefore, if we consider the circumcircle of ΔAEQ, AQ will be a diameter.
Since $QC \perp AQ$, QC is a tangent.
So if we consider the power of C with respect to the circumcircle of ΔAEQ we get

$$|CA|.|CE| = |CQ|^2$$

If we repeat a similar argument on the other side we get

$$|CB|.|CD| = |CP|^2$$

In the quadrilateral AEBD,

$$|\angle AEB| = |\angle ADB| = 90°$$

So these two angles stand on the same arc. Therefore AEBD is cyclic. If we consider the power of C with respect to the circle AEBD, we get

$$|CA|.|CE| = |CB|.|CD|$$

$$\Rightarrow |CQ|^2 = |CP|^2$$

$$|CQ| = |CP|$$